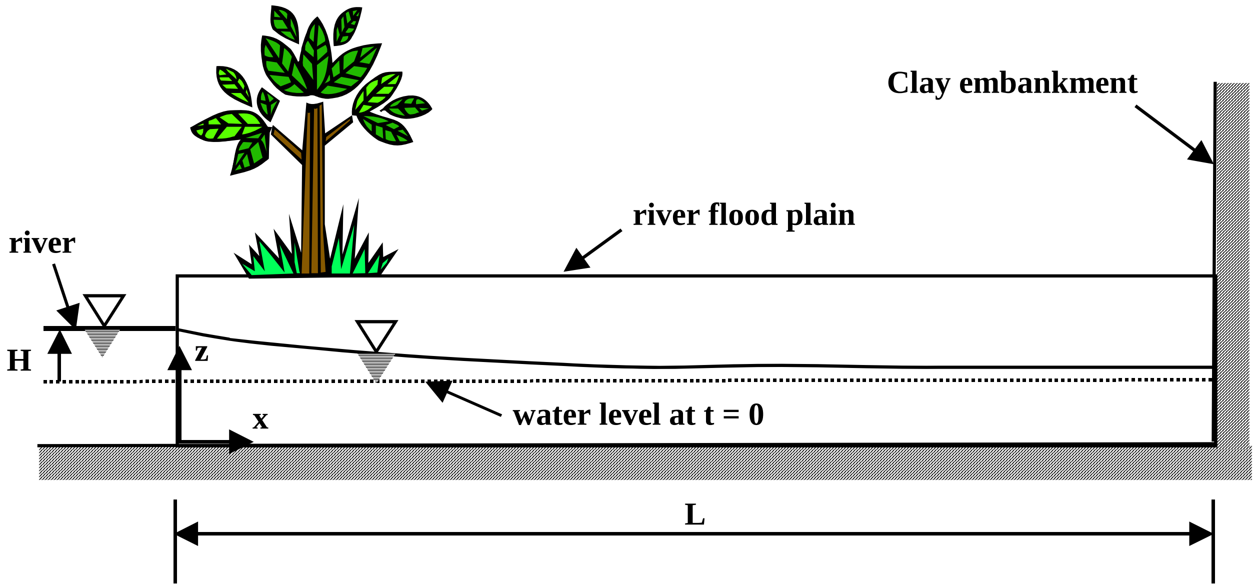
ENCI 302 ENGINEERING MATHEMATICS (Civil)

PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #1



This problem, which is sometimes called the bank storage problem of flood hydrology, describes flow into an aquifer when a river suddenly rises an amount H at t = 0 and is held at this constant elevation for $0 < t < \infty$. Note that the recharge rate, R, is zero for this problem and that $s = s(x,t)$. Furthermore, since the drawdown, s, is positive in the downward direction, calculated values of s can be expected to be negative numbers.

1. Write down all of the equations that are needed for a complete mathematical description of this problem. Remember to show the range over which non-constant independent variables vary for each and every equation.
2. Rewrite in a simpler form the equations obtained in question #1 by choosing an appropriate set of dimensionless variables.
3. Once $s(x,t)$ has been found, then flow into the aquifer is calculated from the following dimensional equation:

$$q(t) = T\frac{\partial s(0,t)}{\partial x} \qquad (0 < t < \infty)$$

   Define a dimensionless variable for q and rewrite this equation in dimensionless form.
4. How would you modify the solution of these equations to obtain the solution when the river level **drops** by an amount H?

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #1 SOLUTION

1. $$T\frac{\partial^2 s}{\partial x^2} = S\frac{\partial s}{\partial t} \qquad (0 < x < L,\ 0 < t < \infty)$$
$$s(0,t) = -H \qquad (0 < t < \infty)$$
$$\frac{\partial s(L,t)}{\partial x} = 0 \qquad (0 < t < \infty)$$
$$s(x,0) = 0 \qquad (0 < x < L)$$

2. $$(s^*, x^*, t^*) = \left(\frac{s}{H}, \frac{x}{L}, \frac{tT}{SL^2}\right)$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} \qquad (0 < x < 1,\ 0 < t < \infty)$$
$$s(0,t) = -1 \qquad (0 < t < \infty)$$
$$\frac{\partial s(1,t)}{\partial x} = 0 \qquad (0 < t < \infty)$$
$$s(x,0) = 0 \qquad (0 < x < 1)$$

3. Define $q^* = q/q_0$ where $q_0 =$ constant. Introduce this into the equation
$$q = T\frac{\partial s(0,t)}{\partial x}$$
to obtain
$$q_0\, q^* = T\frac{s_0}{L}\frac{\partial s^*(0,t^*)}{\partial x^*}$$
Therefore, choose $q_0 = T\dfrac{H}{L}$ so that
$$q^* = \frac{q}{q_0} = \frac{qL}{TH}$$
Then the dimensionless form of the equation becomes:
$$q(t) = \frac{\partial s(0,t)}{\partial x} \qquad (0 < t < \infty)$$
where the asterisk superscript has been omitted for notational convenience.

4. Replace H with – H in the solution.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #2

The following problem is more general than the one described by Eqs. (2.15)-(2.18):

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + f(x) \qquad (0 < x < 1,\ 0 < t < \infty)$$

$$\frac{\partial s(0,t)}{\partial x} = 0 \qquad (0 < t < \infty)$$

$$s(1,t) = 0 \qquad (0 < t < \infty)$$

$$s(x,0) = 0 \qquad (0 < x < 1)$$

where f(x) is a specified function and x is now measured from the embankment toward the river. Use a Fourier series to solve this problem.

Hint:

Obtain this solution by using the following procedure:

1. Choose a Fourier series, with coefficients that are unknown functions of time, that satisfies exactly the boundary condition at x = 0.
2. Choose the "eigenvalues" α_n in the Fourier series so that the boundary condition at x = 1 is satisfied exactly.
3. Substitute the resulting Fourier series into the PDE to obtain an equation containing the unknown coefficients $a_n(t)$.
4. After making sure that the terms in the Fourier series satisfy the conditions required for the validity of Eqs. (3.7)-(3.10), use these orthogonality relationships to obtain a first-order ordinary differential equation for $a_n(t)$.
5. Use the initial condition for s to obtain an initial condition for $a_n(t)$. Then use this initial condition in the solution of the ordinary differential equation for $a_n(t)$ to obtain a solution for $a_n(t)$.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #2 SOLUTION

Since $\partial s(0,t)=0$, we look for a solution in the form

$$s(x,t)=a_0(t)+\sum_{n=1}^{\infty}a_n(t)\cos(\alpha_n x) \qquad (0<x<1,\ 0<t<\infty)$$

The BC at x = 1 requires

$$s(1,t)=a_0(t)+\sum_{n=1}^{\infty}a_n(t)\cos(\alpha_n)=0 \qquad (0<t<\infty)$$

Therefore, we must choose $a_0(t)=0$ and

$$\boxed{\alpha_n=(2n-1)\frac{\pi}{2}}$$

Thus, both BC's are satisfied exactly by the following Fourier series:

$$\boxed{s(x,t)=\sum_{n=1}^{\infty}a_n(t)\cos(\alpha_n x) \qquad (0<x<1,\ 0<t<\infty)}$$

Substitute this Fourier series into the PDE to obtain

$$\sum_{n=1}^{\infty}\left[\frac{da_n(t)}{dt}+\alpha_n^2\ a_n(t)\right]\cos(\alpha_n x)=-f(x) \qquad (0<x<1)$$

Since $\cos(\alpha_n x)$ satisfies conditions that are required for the validity of Eqs. (3.7)-(3.10) in the course reader, multiply the preceding equation by $\cos(\alpha_m x)$ and integrate from x = 0 to x = 1 to obtain

$$\sum_{n=1}^{\infty}\left(\frac{da_n}{dt}+\alpha_n^2\ a_n\right)\int_0^1\cos(\alpha_m x)\cos(\alpha_n x)\,dx=-\int_0^1 f(x)\cos(\alpha_m x)\,dx$$

Now, use the orthogonality relationships give by Eqs. (3.7) and (3.10) to obtain an ODE with $a_m(t)$ as its only unknown:

$$\left(\frac{da_m}{dt}+\alpha_m^2\ a_m\right)\frac{1}{2}=-C_m$$

where

$$\boxed{C_m\equiv\int_0^1 f(x)\cos(\alpha_m x)\,dx}$$

The IC for s requires that $a_m(0)=0$, and the solution of the ODE for $a_m(t)$ that satisfies this initial condition is given by

$$\boxed{a_m(t)=-\frac{2C_m}{\alpha_m^2}\left(1-e^{-t\alpha_m^2}\right)}$$

Replace m with n in this equation and substitute it into the Fourier series to obtain the final result.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #3

Solve the following problem, which provides an example for which $a_0(t) \neq 0$:

$$\frac{\partial s}{\partial t} - \frac{\partial^2 s}{\partial x^2} = -1 \qquad \left(0 < x < \frac{1}{2}\right)$$

$$= 0 \qquad \left(\frac{1}{2} < x < 1\right)$$

$$\frac{\partial s(0,t)}{\partial x} = 0 \qquad (0 < t < \infty)$$

$$\frac{\partial s(1,t)}{\partial x} = 0 \qquad (0 < t < \infty)$$

$$s(x,0) = 0 \qquad (0 < x < 1)$$

Hint:

Obtain this solution by using the following procedure:

1. Choose a Fourier series, with coefficients that are unknown functions of time, that satisfies exactly the boundary condition at x = 0.
2. Choose the "eigenvalues" α_n in the Fourier series so that the boundary condition at x = 1 is satisfied exactly.
3. Substitute the resulting Fourier series into the PDE to obtain an equation containing the unknown coefficients $a_n(t)$.
4. After making sure that the terms in the Fourier series satisfy the conditions required for the validity of Eqs. (3.7)-(3.10), use these orthogonality relationships to obtain a first-order ordinary differential equation for $a_n(t)$.
5. Use the initial condition for s to obtain an initial condition for $a_n(t)$. Then use this initial condition in the solution of the ordinary differential equation for $a_n(t)$ to obtain a solution for $a_n(t)$.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #3 SOLUTION

Since $\partial s(0,t)/\partial x = 0$, we look for a solution in the form

$$s(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(\alpha_n x) \qquad (0 < x < 1,\ 0 < t < \infty)$$

The BC at x = 1 requires

$$\frac{\partial s(1,t)}{\partial x} = 0 - \sum_{n=1}^{\infty} a_n(t)\alpha_n \sin(\alpha_n) = 0 \qquad (0 < t < \infty)$$

Therefore, we must choose

$$\boxed{\alpha_n = n\pi}$$

Thus, both BC's are satisfied exactly by the following Fourier series:

$$\boxed{s(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(\alpha_n x) \qquad (0 < x < 1,\ 0 < t < \infty,\ \alpha_n = n\pi)}$$

Substitute this Fourier series into the PDE to obtain

$$\frac{da_0(t)}{dt} + \sum_{n=1}^{\infty}\left[\frac{da_n(t)}{dt} + \alpha_n^2\ a_n(t)\right]\cos(\alpha_n x) = -1 \qquad \left(0 < x < \frac{1}{2},\ 0 < t < \infty\right)$$

$$= 0 \qquad \left(\frac{1}{2} < x < 1,\ 0 < t < \infty\right)$$

Since $\cos(\alpha_n x)$ satisfies conditions that are required for the validity of Eqs. (3.7)-(3.10) in the course reader, multiply the preceding equation by $\cos(\alpha_m x)$ and integrate from x = 0 to x = 1 to obtain

$$\frac{da_0(t)}{dt}\int_0^1 \cos(\alpha_m x)\ dx + \sum_{n=1}^{\infty}\left(\frac{da_n}{dt} + \alpha_n^2\ a_n\right)\int_0^1 \cos(\alpha_m x)\cos(\alpha_n x)\ dx = -\int_0^{1/2}\cos(\alpha_m x)\ dx$$

Now, use the orthogonality relationships give by Eqs. (3.7) and (3.10) to obtain an ODE with $a_m(t)$ as its only unknown:

$$0 + \left(\frac{da_m}{dt} + \alpha_m^2\ a_m\right)\frac{1}{2} = -\frac{\sin(\alpha_m/2)}{\alpha_m}$$

The IC for s requires that $a_m(0) = 0$, and the solution of the ODE for $a_m(t)$ that satisfies this initial condition is given by

$$\boxed{a_m(t) = -2\frac{\sin(\alpha_m/2)}{\alpha_m^3}\left(1 - e^{-t\alpha_m^2}\right) \qquad (m \ge 1)}$$

To calculate $a_0(t)$, integrate both sides of the equation just below the second circled equation to obtain

$$\frac{da_0(t)}{dt}\int_0^1 dx + \sum_{n=1}^{\infty}\left(\frac{da_n}{dt}+\alpha_n^2\ a_n\right)\int_0^1 \cos(\alpha_n x)\ dx = -\int_0^{1/2} dx$$

Thus, Eqs. (3.9) and (3.11) give the following ODE for $a_0(t)$:

$$\frac{da_0(t)}{dt} = -\frac{1}{2}$$

The IC for s requires that $a_0(0)=0$, and the solution of the ODE for $a_0(t)$ that satisfies this initial condition is given by

.

$$\boxed{a_0(t) = -\frac{1}{2}\ t}$$

Putting the expressions for $a_n(t)$ back into the series expansion for s(x, t) gives the final result.

$$\boxed{s(x,t) = -\frac{1}{2}t - 2\sum_{n=1}^{\infty}\frac{\sin(\alpha_n/2)}{\alpha_n^3}\left(1-e^{-t\alpha_n^2}\right)\cos(\alpha_n x) \qquad (\alpha_n = n\pi)}$$

There must be a nice neat expression for $\sin(\alpha_n/2) \equiv \sin(n\pi/2) = 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ ....$ for n = 1, 2, 3, 4, 5, 6, …., but I don't know what it might be. If you know, then please tell me!

ENCI 302 ENGINEERING MATHEMATICS (Civil)
PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #4

Solve the following problem, which has already been written in dimensionless variables:

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} + R \qquad (0 < x < 1,\ 0 < t < \infty)$$
$$s(0,t) = 0 \qquad (0 < t < \infty)$$
$$s(1,t) = 0 \qquad (0 < t < \infty)$$
$$s(x,0) = -H \qquad (0 < x < 1)$$

where R and H are constants.

Hint:

Obtain this solution by using the following procedure:

1. Choose a Fourier series, with coefficients that are unknown functions of time, that satisfies exactly the boundary condition at x = 0.
2. Choose the "eigenvalues" α_n in the Fourier series so that the boundary condition at x = 1 is satisfied exactly.
3. Substitute the resulting Fourier series into the PDE to obtain an equation containing the unknown coefficients $a_n(t)$.
4. After making sure that the terms in the Fourier series satisfy the conditions required for the validity of Eqs. (3.7)-(3.10), use these orthogonality relationships to obtain a first-order ordinary differential equation for $a_n(t)$.
5. Use the initial condition for s to obtain an initial condition for $a_n(t)$. Then use this initial condition in the solution of the ordinary differential equation for $a_n(t)$ to obtain a solution for $a_n(t)$.

Although you have not made use of this in solving the problem, you should notice that your final result is the superposition of the solution for $R \neq 0$ when H = 0 and the solution for R = 0 when $H \neq 0$.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #4 SOLUTION

The following equation satisfies both homogeneous boundary conditions:

$$s(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(\alpha_n x) \qquad (\alpha_n = n\pi)$$

Substitute this expression for s(x, t) into the PDE to obtain

$$\sum_{n=1}^{\infty} \left(a_n' + \alpha_n^2 a_n\right) \sin(\alpha_n x) = -R \qquad (0 < x < 1)$$

Multiply both sides of the previous equation by $\sin(\alpha_m x)$ and integrate from x = 0 to x = 1 to obtain the following equation:

$$\sum_{n=1}^{\infty} \left(a_n' + \alpha_n^2 a_n\right) \int_0^1 \sin(\alpha_m x) \sin(\alpha_n x)\,dx = -R \int_0^1 \sin(\alpha_m x)\,dx$$

Since the $\sin(\alpha_n x)$ vanishes at both x = 0 and at x = 1, the othogonality conditions given by Eqs. (3.7)-(3.10) are valid. Therefore, evaluation of the integrals gives the following differential equation for $a_n(t)$:

$$a_m' + \alpha_n^2 a_m = 2R \frac{(-1)^m - 1}{\alpha_m}$$

in which the right side is just a constant that depends upon α_m. This equation requires an initial condition, which is found by substituting the Fourier series expression for s(x, t) into the initial condition s(x, 0) = -H to obtain

$$-H = \sum_{n=1}^{\infty} a_n(0) \sin(\alpha_n x) \qquad (0 < x < 1)$$

Multiply both sides of the previous equation by $\sin(\alpha_m x)$ and integrate from x = 0 to x = 1 to obtain the following equation:

$$-H \int_0^1 \sin(\alpha_m x)\,dx = \sum_{n=1}^{\infty} a_n(0) \int_0^1 \sin(\alpha_m) \sin(\alpha_n x)\,dx$$

Since the $\sin(\alpha_n x)$ vanishes at both x = 0 and at x = 1, the othogonality conditions given by Eqs. (3.7)-(3.10) are valid. Therefore, evaluation of the integrals gives the following initial condition for $a_n(t)$:

$$a_m(0) = 2H \frac{(-1)^m - 1}{\alpha_m}$$

Solution of the problem for $a_m(t)$ fives the following result:

$$a_m(t) = 2H \frac{(-1)^m - 1}{\alpha_m} e^{-t\alpha_m^2} + 2R \frac{(-1)^m - 1}{\alpha_m^3}\left(1 - e^{-t\alpha_m^2}\right)$$

Substituting this result in the Fourier series expansion for s(x, t) gives the following solution:

$$s(x,t) = 2H \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\alpha_n} e^{-t\alpha_n^2} \sin(\alpha_n x) + 2R \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\alpha_n^3}\left(1 - e^{-t\alpha_n^2}\right) \sin(\alpha_n x) \qquad (\alpha_n = n\pi)$$

This final result is the superposition of the solution for $R \neq 0$ when H = 0 and the solution for R = 0 when $H \neq 0$.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #5

The following is an example of a problem with inhomogeneous boundary conditions:

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial s}{\partial t} \qquad (0 < x < 1,\ 0 < t < \infty)$$

$$\frac{\partial s(0,t)}{\partial x} = 1 \qquad (0 < t < \infty)$$

$$s(1,t) = H \qquad (0 < t < \infty)$$

$$s(x,0) = 0 \qquad (0 < x < 1)$$

Solve this problem by adopting the following procedure:

1. Since the method of solution that you have been taught works best when both boundary conditions are homogeneous, define a new dependent variable, $\varphi(x,t)$, with the equation
$$s(x,t) = a + bx + \varphi(x,t)$$
Then choose the constants a and b so that $\varphi(x,t)$ satisfies homogeneous boundary conditions.
2. Write down the resulting problem for $\varphi(x,t)$.
3. Use an appropriate Fourier series to solve for $\varphi(x,t)$.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #5 SOLUTION

Set $s(x,t) = a + bx + \varphi(x,t)$ and obtain two equations for a and b from the inhomogeneous BC's for s(x, t).

$$1 = \frac{\partial s(0,t)}{\partial x} = b + \frac{\partial \varphi(0,t)}{\partial x} = b \qquad \left(\text{Since } \frac{\partial \varphi(0,t)}{\partial x} = 0\right)$$

$$H = s(1,t) = a + b + \varphi(1,t) = a + b \qquad \left(\text{Since } \varphi(1,t) = 0\right)$$

These equations give b = 1 and a = H-1, and the equation for s(x, t) now becomes

$$\boxed{s(x,t) = H - 1 + x + \varphi(x,t)}$$

Substitute this expression for s(x, t) into the problem for s(x, t) to obtain the following problem for $\varphi(x,t)$:

$$\boxed{\begin{array}{ll} \dfrac{\partial^2 \varphi}{\partial x^2} = \dfrac{\partial \varphi}{\partial t} & (0 < x < 1,\ 0 < t < \infty) \\ \dfrac{\partial \varphi(0,t)}{\partial x} = 0 & (0 < t < \infty) \\ \varphi(1,t) = 0 & (0 < t < \infty) \\ \varphi(x,0) = 1 - H - x & (0 < x < 1) \end{array}}$$

Look for a solution for φ in the following form:

$$\varphi(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos(\alpha_n x)$$

The homogeneous BC's for φ will be satisfied if $a_0(t) = 0$ and if α_n is given by the odd multiples of $\pi/2$. Therefore, we have

$$\boxed{\varphi(x,t) = \sum_{n=1}^{\infty} a_n(t) \cos(\alpha_n x) \qquad \left[\alpha_n \equiv (2n-1)\frac{\pi}{2}\right]}$$

Substitution into the PDE gives

$$\sum_{n=1}^{\infty} \left(a_n' + \alpha_n^2 a_n\right) \cos(\alpha_n x) = 0 \qquad (0 < x < 1)$$

which will be satisfied if $a_m(t)$ is a solution of the following ODE:

$$a_m' + \alpha_m^2 \, a_m = 0$$

An initial condition for $a_n(t)$ is found from the initial condition for $\varphi(x,t)$:

$$1 - H - x = \varphi(x,0) = \sum_{n=1}^{\infty} a_n(0) \cos(\alpha_n x)$$

Use of the orthogonality conditions to pick off $a_m(0)$ gives

$$a_m(0) = 2\int_0^1 (1 - H - x)\cos(\alpha_m x)\,dx = \frac{2}{\alpha_m^2} + 2H\frac{(-1)^m}{\alpha_m}$$

Therefore, solution for $a_m(t)$ gives the result

$$\boxed{a_m(t) = \left[\frac{2}{\alpha_m^2} + 2H\frac{(-1)^m}{\alpha_m}\right] e^{-t\alpha_m^2}}$$

Inserting the previous expression for $a_m(t)$ in the expression for $\varphi(x,t)$, and inserting that result in the first circled equation gives the following result for s(x, t):

$$\boxed{s(x,t) = H - 1 + x + \sum_{n=1}^{\infty}\left[\frac{2}{\alpha_n^2} + 2H\frac{(-1)^n}{\alpha_n}\right] e^{-t\alpha_n^2}\cos(\alpha_n x) \qquad \left[\alpha_n \equiv (2n-1)\frac{\pi}{2}\right]}$$

ENCI 302 ENGINEERING MATHEMATICS (Civil)
PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #6

An elastic rod is subjected to a longitudinal force that has a sinusoidal variation with time. The partial differential equation and boundary conditions that describe the "steady-state" solution follow:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad (0 < x < L,\ -\infty < t < \infty)$$

$$u(0,t) = u_0 \sin(\omega t) \qquad (-\infty < t < \infty,\ u_0 \text{ and } \omega \text{ are constants})$$

$$u(L,t) = 0 \qquad (-\infty < t < \infty)$$

Answer the following questions in regard to this problem:

1. Explain what is meant by a steady-state solution in this context.

2. Calculate the steady-state solution.

3. Calculate values of ω for which resonance will occur.

4. Use the trigonometric identities

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

   to rewrite the solution from question 2 in the form $u(x,t) = f(x-ct) + g(x+ct)$.

5. Explain the significance of the answer to question 4.

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PARTIAL DIFFERENTIAL EQUATIONS TUTORIAL #6 SOLUTION

1. The term "steady-state solution" in this context means the solution that is left after a sufficiently long time has elapsed for the effect of **initial conditions** to become "forgotten". In a physical problem this "forgetting process" is the result of small amounts of damping or friction in a system. This damping can often be neglected over small periods of time, but over longer periods of time the accumulated effect of even a small amount of damping in a system causes the same steady-state solution to be approached for any set of initial conditions. This is the reason why I specify a time domain $(-\infty < t < \infty)$ for a steady-state problem rather than $(0 < t < \infty)$, which I use for a problem in which initial conditions are specified.

2. The presence of the sinusoidal term $\sin(\omega t)$ and the fact that only time derivatives of even order appear in the partial differential equation suggest that I look for a solution in the form

$$u(x,t) = u_0 \ \varphi(x) \ \sin(\omega t) \tag{1}$$

Insert Eq. (1) into the partial differential equation to obtain the following ordinary differential equation for $\varphi(x)$:

$$\varphi''(x) + \left(\frac{\omega}{c}\right)^2 \varphi(x) = 0 \tag{2}$$

Eq. (2) is a second-order ordinary differential equation, which means that it will have two integration constants in its general solution and, therefore, will require two boundary conditions for its unique solution. Insert Eq. (1) into the two boundary conditions for $u(x,t)$ to obtain boundary conditions for $\varphi(x)$.

$$\varphi(0) = 1 \tag{3}$$

$$\varphi(L) = 0 \tag{4}$$

The solution of Eqs. (2)-(4) is given by

$$\varphi(x) = \cos(\omega x / c) - \frac{\cos(\omega L / c)}{\sin(\omega L / c)} \sin(\omega x / c) \tag{5}$$

which can be manipulated into the form

$$\varphi(x) = \frac{\sin(\omega L / c)\cos(\omega x / c) - \sin(\omega x / c)\cos(\omega L / c)}{\sin(\omega L / c)} = \frac{\sin \omega (L - x) / c}{\sin(\omega L / c)} \tag{6}$$

Therefore, inserting Eq. (6) into Eq. (1) gives the following steady-state solution:

$$u(x,t) = u_0 \ \frac{\sin \omega (L - x) / c}{\sin(\omega L / c)} \ \sin(\omega t) \qquad (0 < x < L, \ -\infty < t < \infty) \tag{7}$$

3. Eq. (7) shows that resonance occurs when $\sin(\omega L / c) = 0$, which gives the following resonance frequencies for ω:

$$\omega L / c = n\pi \ \text{for} \ n = 1, 2, 3, 4, ...$$

4. Inserting Eq. (5) into Eq. (1) gives

$$u(x,t) = u_0 \cos(\omega x/c)\sin(\omega t) - u_0 \frac{\cos(\omega L/c)}{\sin(\omega L/c)}\sin(\omega x/c)\sin(\omega t) \tag{8}$$

However, the trigonometric identities written in the problem statement give

$$\cos(\omega x/c)\sin(\omega t) = \frac{1}{2}\left[\sin(\omega t + \omega x/c) + \sin(\omega t - \omega x/c)\right] = \frac{1}{2}\left[\sin\frac{\omega}{c}(x+ct) - \sin\frac{\omega}{c}(x-ct)\right] \tag{9}$$

and

$$\sin(\omega x/c)\sin(\omega t) = \frac{1}{2}\left[\cos(\omega x/c - \omega t) - \cos(\omega x/c + \omega t)\right] = \frac{1}{2}\left[\cos\frac{\omega}{c}(x-ct) - \cos\frac{\omega}{c}(x+ct)\right] \tag{10}$$

where the identity $\sin(-x) = -\sin(x)$ has also been used in Eq. (9). Inserting Eqs. (9) and (10) in Eq. (8) gives the final result

$$u(x,t) = f(x-ct) + g(x+ct) \tag{11}$$

where

$$f(x-ct) = -\frac{u_0}{2}\sin\frac{\omega}{c}(x-ct) - \frac{u_0}{2}\frac{\cos(\omega L/c)}{\sin(\omega L/c)}\cos\frac{\omega}{c}(x-ct) \tag{12}$$

$$g(x+ct) = \frac{u_0}{2}\sin\frac{\omega}{c}(x+ct) + \frac{u_0}{2}\frac{\cos(\omega L/c)}{\sin(\omega L/c)}\cos\frac{\omega}{c}(x+ct) \tag{13}$$

5. Eq. (11) gives the general solution of the hyperbolic partial differential equation

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

where $f(x-ct)$ is the equation of a wave with constant form travelling in the positive x direction with a speed or celerity c and $g(x+ct)$ is the equation of a wave with constant form travelling in the negative y direction with a speed or celerity c.